RESEARCH STATEMENT

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1. Overview

My mathematical research is in algebraic and enumerative combinatorics, more specifically, the asymmetric exclusion process, orthogonal polynomials, and Ehrhart theory. In particular, I am interested in giving combinatorial formulas for polynomials with positive coefficients. Explaining the positivity of such polynomials is an interesting problem by itself, and giving combinatorial formulas is useful as they provide a fast and compact way to compute those polynomials.

The asymmetric exclusion process (ASEP) is an important model from statistical mechanics which describes a system of particles on a lattice hopping left and right. This process was introduced in the 1970's independently in the context of biology and in mathematics. Since then this model was studied extensively in various fields for number of reasons. First, the ASEP exhibits a rich phenomenology and has many applications in a broad range including protein synthesis, traffic flow, formation shocks, surface growth, and sequence alignments. Second the study of the ASEP allows various mathematical approaches, for example, Bethe Ansatz, quadratic algebras, combinatorics, orthogonal polynomials, random matrices, stochastic differential equations and hydrodynamic limits.

The totally asymmetric exlusion process (TASEP) on a ring is a Markov chain on a periodic one dimensional lattice of length N where each lattice site can be either occupied by a particle or empty. A particle can hop to its right (when it is empty) with rate 1 (see Figure 1). The number of particles is preserved so there are a total of $\binom{N}{k}$ possible states in this Markov chain where k is the number of particles. There is a multi-species generalization of the TASEP by allowing several types of particles. The important question of the TASEP is to understand the steady state probabilities and combinatorial approach has been successful. The combinatorial object multiline queue was introduced in [18] to describe the steady state probabilities

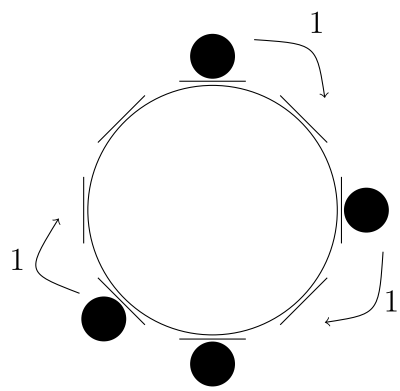

Figure 1. The figure shows transition rates of the TASEP on a ring

of the multispecies TASEP on a ring. There has been studies on a multiline queue in [15, 16] as it has a rich combinatorial structure. Our work in progress shows that many steady state probabilities are proportional to product of Schubert polynomials [12].

The ASEP on a line is a Markov chain on a one dimensional lattice of length N with open boundaries. A particle can hop to the right with rate 1 and can hop to the left with rate q, as long as the neighboring site is empty. And on each boundary, a particle can enter from the left or right with rate α or δ respectively and can exit to the left or right with rate γ or β respectively (see Figure 2). The number of particles

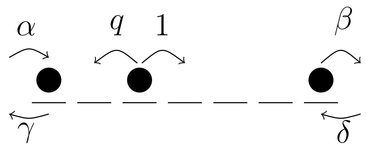

FIGURE 2. The figure shows transition rates of the ASEP on a line

is not preserved so there are a total of 2^N possible states in this Markov chain. Papers of Sasamoto [7] and subsequently Uchiyama, Sasamoto, and Wadati [8] revealed a surprising connection between the ASEP on a line and orthogonal polynomials, in particular the Askey-Wilson polynomials which lie in a top hierarchy of (basic) hypergeometric orthogonal polynomials in the sense that all other polynomials in this hierarchy are limiting cases or specializations of the Askey-Wilson polynomials. Orthogonal polynomials have beautiful combinatorial properties [19, 20, 21]. I have given combinatorial formulas for the Al-Salam-Chihara polynomials which are related to the ASEP when $\gamma = \delta = 0$ and am working on generalization to the Askey-Wilson polynomials.

In 1960's, Ehrhart introduced Ehrhart polynomials and Ehrhart series to study number of lattice points inside polytopes. Since then, there has been a lot of study on Ehrhart polynomials and Ehrhart series of many well-known polytopes. The (k, n)*-th hypersimplex* $\Delta_{k,n}$ is a lattice polytope inside $\mathbb{R}^n$ whose vertices are (0,1)-vectors with exactly k 1's. The hypersimplex can be found in several algebraic and geometric contexts, for example, as a moment polytope for the torus action on the Grassmannian, or as a weight polytope for the fundamental representation of GL_n. I gave the first combinatorial formula for the Ehrhart series of the hypersimplex [4], proving a conjecture of Early [1]. Later, Stapledon introduced equivariant version of Ehrhart polynomials and Ehrhart series to study group action on lattice points inside polytopes [3]. One of the questions I am working on is to combinatorially understand equivariant Ehrhart series of a polytope, for example, the hypersimplex $\Delta_{k,n}$.

The progress and the direction of the projects will be explained below.

2. Inhomogeneous totally asymmetric exclusion process

The inhomogeneous multispecies TASEP on a ring is a Markov chain on a periodic lattice of length N where each lattice site is occupied by positive integers. This Markov chain is indexed by $m = (m_1, m_2, \cdots)$ such that $\sum m_i = N$ where m_i is the number of i's on a lattice. There are a total of $\binom{N}{m_1, m_2, \cdots}$ possible states in this case. The adjacent integers i and j (i is on the left of j) can swap their positions with a rate $r_{i,j}$ given as follows

$$r_{i,j} = \begin{cases} x_i & \text{if } i < j \\ 0 & \text{otherwise} \end{cases} .$$

When each $m_i = 1$, the possible states are permutations. In [11], Lam and Williams

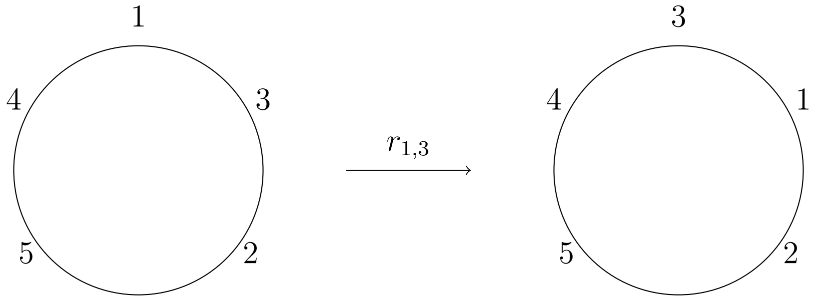

FIGURE 3. The figure shows a transitions rate of the inhomogeneous multispecies TASEP on a ring.

conjectured that each steady state probability is proportional to a positive linear combination of Schubert polynomials. Subsequently, a combinatorial formula for the steady state probability was given in terms of objects called multiline queues [18, 9, 10]. In particular, the steady state probability for the state w is given as a weighted sum over multiline queues of type w. However, the conjecture about Schubert polynomials remained open.

In [22], Cantini introduced a z-deformed steady state probability for the spectral parameters $z_1, z_2, \cdots$ which recovers the (usual) steady state probability when specialized to $z_i = \infty$. He gave an explicit formula for z-deformed steady state probability of a few states, explaining the appearance of Schubert polynomials in those cases.

In a joint with Lauren Williams [12], we showed that when w is an inverse of Grassmann permutation (a permutation with at most one descent), there is a bijection between multiline queues of type w and monomials of a corresponding Schubert polynomial. This is the first work to directly connect multiline queues and Schubert polynomials. We also introduced a special subset $SS(n, k)$ of S_n and gave an explicit formula for z-deformed stationary distribution of $w \in SS(n, k)$ thereby generalizing Cantini's work. As a corollary we have the following.

Theorem 2.1 ([12]). *For $w \in SS(n, k)$, the stationary distribution of the state w is proportional to a product of k Schubert polynomials.*

It is a future task to understand the geometric interpretation of the steady state probabilities.

3. Orthogonal polynomials

We say that a family $(p_n(x))_{n\geq 0}$ of polynomials in one variable is *orthogonal* if the degree of $p_n(x)$ is n and they are orthogonal with respect to a certain measure ω, that is

$$\int p_n(x)p_m(x)d\omega = 0, \text{ for } m \neq n .$$

And the *N-th moments* μ_N of $(p_n(x))_{n\geq 0}$ are defined as $\mu_N = \int x^N d\omega,\ \text{for } N \geq 0$. The most widely used orthogonal polynomials are hypergeometric orthogonal polynomials, which include the Hermite polynomials, the Laguerre polynomials, and the Jacobi polynomials.

The Askey-Wilson polynomials are a family of orthogonal polynomials which include many of the other orthogonal polynomials as special or limiting cases. They lie at the top of the hierarchy in the Askey scheme. Surprisingly, in [8], the moments of this family are connected to the steady state probability of the ASEP on a line. Later in [6], Corteel and Williams gave a combinatorial formula for the steady state probabilities in particular showing that the moments of the Askey-Wilson polynomials are polynomials in $\alpha, \beta, \gamma, \delta$ and q with positive coefficients. Conjecturally, the coefficients of Askey-Wilson polynomials are polynomials in $\alpha, \beta, \gamma, \delta$ and q with positive coefficients. Trying to explain this positivity and give a combinatorial formula was the start of my work in this area.

In [13], I studied the coefficients of the Al-Salam-Chihara polynomials, which are obtained as the specialization of the Askey-Wilson polynomials at $\gamma = \delta = 0$. I gave a combinatorial formula for those coefficients, which explains positivity in the case $\gamma = \delta = 0$. To do this, I introduced a generalized q-binomial coefficient $M_n^{\mu}(b)$. Here n and b are non-negative integers and μ is a weakly increasing composition of length a. It is a polynomial in q and α that recovers ordinary q binomial coefficient $\binom{n+a+b}{b}_q$ when $\alpha = 0$. The construction of $M_n^{\mu}(b)$ was motivated by my bijective proof of the identity $\binom{n+a}{a}_q \binom{n+a+b}{b}_q = \binom{n+a+b}{a}_q \binom{n+b}{b}_q$. I also showed that the well known identity (3.1)

$$\binom{n+a+b+1}{b}_q = q^{n+a+1}\binom{n+a+b}{b-1}_q + \binom{n+a+b}{b}_q \tag{3.1}$$

lifts to (3.2) (Lemma 2.11 in [13])

$$M_{n+1}^{\mu}(b) = (q^{n+a+b} + [n+a+b]_q \alpha) M_{n+1}^{\mu}(b-1) + M_n^{\mu-1}(b). \tag{3.2}$$

So far, there seems to be no previous work that can be related to a generalized q-binomial coefficient. It would be interesting to study this and find more applications. My main result in [13] expresses the coefficients of the (transformed) Al-Salam-Chihara polynomials in terms of the generalized q-binomial coefficients.

Theorem 3.1 ([13]). *The coefficient of x^n in the (transformed) Al-Salam-Chiara polynomial $\tilde{p}_{n+k}(x)$ is given by*

$$[x^n]\hat{p}_{n+k}(x) = \sum_{a+b=k} \Big(\sum_{\substack{\mu=(\mu_1,\cdots,\mu_a) \\ 0\leq\mu_1\leq\cdots\leq\mu_a\leq n}} (\xi\alpha)^a \beta^b X_\mu M_n^{\mu}(b) \Big),$$

where $X_\mu = \prod_{i=1}^{a} (q^{\mu_i+i-1} + [\mu_i + i - 1]_q \beta)$.

Theorem 3.1 makes clear that the coefficients of the (transformed) Al-Salam-Chihara polynomials are polynomials with positive coefficients.

In [14], I gave a conjectural formula for the coefficients of the Askey-Wilson polynomials. Proving this formula will explain the positivity of the coefficients of the Askey-Wilson polynomials.

The guiding problem of this project is to understand the minors of the coefficient matrix $G = (g_{n,i})_{n,i}$ where $g_{n,i}$ is the coefficient of x^i of the Askey-Wilson polynomial $p_n(x)$ if $i \leq n$, otherwise zero.

$$G = \begin{bmatrix} 1 & g_{1,0} & g_{2,0} & g_{3,0} \cdots \\ 0 & 1 & g_{2,1} & g_{3,1} \cdots \\ 0 & 0 & 1 & g_{3,2} \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The minors of G are also conjectured to be polynomials with positive coefficients. Unfortunately, techniques showing a positivity of minors in the literature (for example, Lindström-Gessel-Viennot lemma) does not apply to the coefficient matrix G. In [5], a combinatorial formula for some minors was given by relating it to the stationary distributions of 2-species ASEP. And in [13], I proved positivity for some 2 by 2 minors when $\gamma = \delta = 0$ using generalized q-binomial coefficients.

Theorem 3.2 ([13]). *Let $g_{n+k,n}$ be the coefficient of x^n of the (transformed) Al-Salam-Chihara polynomial $\tilde{p}_{n+k}(x)$. Then*

$$(g_{n+a+b,n+a}g_{n+a,n} - g_{n+a+b,n})$$

is a polynomial with positive coefficients.

4. Equivariant Ehrhart theory of the hypersimpelx

For an n-dimensional lattice polytope $\mathcal{P} \subset \mathbb{R}^N$, it is well known from Ehrhart theory that the map $r \to |r\mathcal{P} \cap \mathbb{Z}^N|$ is a polynomial function in r of degree n, which we call *Ehrhart polynomial*, and the corresponding *Ehrhart series* $\sum_{r=0}^{\infty} |r\mathcal{P} \cap \mathbb{Z}^N| t^r$ is a rational function of the form

$$\sum_{r=0}^{\infty} |r\mathcal{P} \cap \mathbb{Z}^N| t^r = \frac{h^*(t)}{(1-t)^{n+1}},$$

such that $h^*(t)$ is a polynomial of degree $\leq n$ (see [25]). Define h^*_d to be the coefficient of t^d in $h^*(t)$. The vector $(h^*_0, \cdots, h^*_n)$ is called the Ehrhart h^*-vector of $\mathcal{P}$. In [2], Stanley proved that the h^*-vector of a lattice polytope always consists of non-negative integers, so it became an interesting question to find a combinatorial interpretation of h^*-vectors for various polytopes.

In [1], Early conjectured a combinatorial interpretation of the h^*-vector of the hypersimplex $\Delta_{k,n}$ in terms of decorated ordered set partitions. A *decorated ordered set partition* $((L_1)_{l_1}, \cdots, (L_m)_{l_m})$ of type (k, n) consists of an ordered partition $(L_1, \cdots, L_m)$ of $\{1, 2, ..., n\}$ and an m-tuple $(l_1, \cdots, l_m) \in \mathbb{Z}^m$ such that $l_1 + \cdots + l_m = k$ and $l_i \geq 1$. A decorated ordered set partition is called *hypersimplicial* if it satisfies $1 \leq l_i \leq |L_i| - 1$ for all i, and one can define a natural statistic called *winding number* for them. In [4], I proved the conjecture of Early.

Theorem 4.1 ([4]). *The h^*-vector $(h^*_0, h^*_1, \cdots)$ of the hypersimplex $\Delta_{k,n}$ has the property that h^*_d equals the number of hypersimplicial decorated ordered set partitions of type (k, n) with winding number d.*

Theorem 4.1 was the first result to give a combinatorial interpretation of the h^*-vector of the hypersimplex. The proof is purely combinatorial so it is still an interesting question to ask for a geometric explanation of Theorem 4.1.

In [3], Stapledon introduced the G-equivariant h^*-vector of a lattice polytope $\mathcal{P}$. Here G is a finite group acting on a lattice M by $\rho : G \to GL(M)$ and $\mathcal{P}$ is a polytope that is invariant under the G-action. Then consider $\chi_{m\mathcal{P}}$, the character of the permutation representation of G on lattice points inside $m\mathcal{P}$. Writing

$$\sum_{m\geq 0}(\chi_{m\mathcal{P}})t^m = \frac{\phi(t)}{(1-t)det(I-\rho t)},$$

we say that the coefficient vector of $\phi(t)$ is the *G-equivariant h^*-vector* of $\mathcal{P}$. Note that the G-equivariant h^*-vector consists of characters of G and plugging in identity element of G to those characters recovers the usual h^*-vector.

Consider the action of $\mathbb{Z}/n\mathbb{Z}$ on the lattice $\mathbb{Z}^n$ which permutes coordinates. The hypersimplex $\Delta_{k,n}$ is invariant under this action so we can consider the $\mathbb{Z}/n\mathbb{Z}$-equivariant h^*-vector. In [17], I showed that $\mathbb{Z}/n\mathbb{Z}$-equivariant h^*-vector of the hypersimplex comes from a natural $\mathbb{Z}/n\mathbb{Z}$ action on decorated ordered set partitions. It is a future task to extend this result to S_n. In [17], I showed that the S_n-equivariant h^*-vector of the hypersimplex consists of effective characters, so it is an interesting question to construct representations that those characters come from.

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